

# Locally Connected HL Compacta\*

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## Abstract

It is consistent with  $\text{MA} + \neg\text{CH}$  that there is a locally connected hereditarily Lindelöf compact space which is not metrizable.

## 1 Introduction

All spaces discussed in this paper are assumed to be Hausdorff. A question attributed in 1982 by Nyikos [8] to M. E. Rudin asks whether  $\text{MA} + \neg\text{CH}$  implies that every locally connected hereditarily Lindelöf (HL) compact space is metrizable (equivalently, second countable); see Gruenhage [5] for further discussion. Filippov [4] had constructed such a space in 1969 under CH, and his space is also hereditarily separable (HS). Since Filippov used a Luzin set in his construction, and  $\text{MA} + \neg\text{CH}$  implies that there are no Luzin sets, it might have been hoped that  $\text{MA} + \neg\text{CH}$  refutes the existence of such a space, but that turns out to be false; we shall show in Section 3:

**Theorem 1.1** *It is consistent with  $\text{MA} + 2^{\aleph_0} = \aleph_2$  that there is a non-metrizable locally connected compactum which is both HS and HL.*

Our proof shows in ZFC that the Filippov construction succeeds provided that there is a *weakly Luzin set*; details are in Section 2. Weakly Luzin sets are related to entangled sets, and our proof of Theorem 1.1 shows that weakly Luzin sets are consistent with  $\text{MA} + 2^{\aleph_0} = \aleph_2$ . We can show that PFA refutes spaces which are “like” the Filippov space (see Section 4), but we do not know whether PFA refutes all non-metrizable locally connected HL compacta.

The Filippov space may be viewed as a connected version of the double arrow space  $D$ , which was described in 1929 by Alexandroff and Urysohn [2]. This is a ZFC

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example of a non-metrizable compactum which is both HS and HL, but it is totally disconnected. The cone over  $D$  yields a connected example, but this is not locally connected.

$D$  is constructed from  $[0, 1]$  by replacing the points of  $(0, 1)$  by neighboring pairs of points. To construct the Filippov space, start with  $[0, 1]^2$ , choose a set  $E \subseteq (0, 1)^2$ , and replace the points of  $E$  by circles, obtaining a space  $\Phi_E$ . This  $\Phi_E$  is compact and locally connected.  $\Phi_E$  is metrizable iff  $E$  is countable. Furthermore, if  $E$  is a Luzin set, then, as Filippov showed,  $\Phi_E$  is HL, and a similar proof shows that  $\Phi_E$  is HS as well.

Actually, by Juhász [7] and Szentmiklóssy [9], HS and HL are equivalent for compacta under  $\text{MA}(\aleph_1)$ , but that result is not needed here. We shall show in ZFC (Theorem 2.5) that  $\Phi_E$  is HS iff  $\Phi_E$  is HL iff  $E$  is weakly Luzin.

## 2 Weakly Luzin Sets

We begin by describing Filippov's example [4]. We start with  $[0, 1]^n$  (where  $1 \leq n < \omega$ ), rather than  $[0, 1]^2$ , to show that the construction does not depend on accidental features of two-dimensional geometry. As usual,  $S^{n-1} \subset \mathbb{R}^n$  denotes the unit sphere, and  $\|x\|$  denotes the length of  $x \in \mathbb{R}^n$ , using the standard Pythagorean metric. Given  $E \subseteq (0, 1)^n$ , we shall obtain the space  $\Phi_E$  by replacing all points in  $E$  by  $(n-1)$ -spheres and leaving the points in  $[0, 1]^n \setminus E$  alone.

**Definition 2.1**  $\rho : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$  is the perpendicular retraction:  $\rho(x) = x/\|x\|$ .

So,  $\rho(y-x)$  may be viewed as the direction from  $x$  to  $y$ .

**Definition 2.2** Fix  $E \subseteq (0, 1)^n$  and let  $E' = [0, 1]^n \setminus E$ . The Filippov space  $\Phi_E$ , as a set, is  $(E \times S^{n-1}) \cup E'$ . Define  $\pi = \pi_E : \Phi_E \rightarrow [0, 1]^n$  so that  $\pi(x, w) = x$  for  $(x, w) \in E \times S^{n-1}$ , and  $\pi(x) = x$  for  $x \in E'$ . For  $\varepsilon > 0$ , define, for  $x \in E'$ :

$$B(x, \varepsilon) = \{p \in \Phi_E : \|\pi(p) - x\| < \varepsilon\} \quad ,$$

and define, for  $x \in E$  and  $W$  an open subset of  $S^{n-1}$ :

$$B(x, W, \varepsilon) = \{x\} \times W \cup \{p \in \Phi_E : 0 < \|\pi(p) - x\| < \varepsilon \text{ \& } \rho(\pi(p) - x) \in W\} \quad .$$

Give  $\Phi_E$  the topology which has all the sets  $B(x, \varepsilon)$  and  $B(x, W, \varepsilon)$  as a base.

**Lemma 2.3** For each  $E \subseteq (0, 1)^n$ :  $\Phi_E$  is compact and first countable.  $\pi_E$  is a continuous irreducible map from  $\Phi_E$  onto  $[0, 1]^n$ .  $\Phi_E$  is metrizable iff  $E$  is countable. If  $n \geq 2$ , then  $\Phi_E$  is connected and locally connected, and  $\pi_E$  is monotone.

The proof of this last sentence uses the connectedness of  $S^{n-1}$ . When  $n = 1$ ,  $S^0 = \{\pm 1\}$ , and  $\Phi_E$  is just the double arrow space obtained by doubling the points of  $E$ , so  $\Phi_E$  is always HS and HL. When  $n > 1$ , the argument of Filippov shows that  $\Phi_E$  is HL if  $E$  is a Luzin set, but actually something weaker than Luzin suffices:

**Definition 2.4** For  $1 \leq n < \omega$ :

- ✦ If  $T \subseteq \mathbb{R}^n$ , then  $T^* = \{x - y : x, y \in T \text{ \& } x \neq y\}$
- ✦  $T \subseteq \mathbb{R}^n$  is *skinny* iff  $\text{cl}(\rho(T^*)) \neq S^{n-1}$ .
- ✦  $E \subseteq \mathbb{R}^n$  is a *weakly Luzin set* iff  $E$  is uncountable and every skinny subset of  $E$  is countable.

Every subset of a skinny set is skinny, and  $T$  is skinny iff  $\overline{T}$  is skinny. Each skinny set is nowhere dense, so every Luzin set is weakly Luzin. When  $n = 1$ ,  $T$  is skinny iff  $|T| \leq 1$ , every uncountable set is weakly Luzin, and the proof of the following theorem reduces to the usual proof that the double arrow space is HS and HL.

When  $n > 1$ : Under CH, it is easy to construct a weakly Luzin set which is not Luzin. PFA implies that there are no weakly Luzin sets. We shall show in Section 3 that a weakly Luzin set is consistent with  $\text{MA} + \mathfrak{c} = \aleph_2$ . Clearly, if there is a weakly Luzin set in  $\mathbb{R}^n$ , then there is one in  $(0, 1)^n$ .

**Theorem 2.5** For  $n \geq 1$  and uncountable  $E \subseteq (0, 1)^n$ , the following are equivalent:

1.  $E$  is weakly Luzin.
2.  $\Phi_E$  is HS.
3.  $\Phi_E$  is HL.
4.  $\Phi_E$  has no uncountable discrete subsets.

**Proof.** For (4)  $\rightarrow$  (1): If  $E$  is not weakly Luzin, fix an uncountable skinny  $T \subseteq E$ . Let  $W = S^{n-1} \setminus \text{cl}(\rho(T^*))$ , and fix  $w \in W$ . Then  $\{(x, w) : x \in T\} \subset \Phi_E$  is discrete.

Since (2)  $\rightarrow$  (4) and (3)  $\rightarrow$  (4) are obvious, it is sufficient to prove (1)  $\rightarrow$  (2) and (1)  $\rightarrow$  (3). So, assume (1), and let  $\langle p_\alpha : \alpha < \omega_1 \rangle$  be an  $\omega_1$ -sequence of distinct points from  $\Phi_E$ ; we show that it can be neither left separated nor right separated. To do this, fix an open neighborhood  $N_\alpha$  of  $p_\alpha$  for each  $\alpha$ ; we find  $\alpha < \beta < \gamma$  such that  $p_\beta \in N_\alpha$  and  $p_\beta \in N_\gamma$ . This is trivial if  $\aleph_1$  of the  $\pi(p_\alpha)$  lie in  $E'$ , or if  $\aleph_1$  of the  $\pi(p_\alpha)$  are the same point of  $E$ . So, thinning the sequence (discarding some points), and shrinking the neighborhoods (replacing them by smaller ones), we may assume that each  $p_\alpha = (x_\alpha, w_\alpha) \in E \times S^{n-1}$  and that  $N_\alpha = B(x_\alpha, W, \varepsilon)$ , where the  $x_\alpha$  are distinct points in  $E$ ,  $W$  is open in  $S^{n-1}$ , and each  $w_\alpha \in W$ . Let  $T = \{x_\alpha : \alpha < \omega_1\}$ . Thinning further, we may assume that  $\text{diam}(T) < \varepsilon$ , so that  $p_\beta \in N_\alpha$  iff  $\rho(x_\beta - x_\alpha) \in W$ . Thinning again, we may assume that every point of  $T$  is a condensation point of  $T$ . Since  $E$  is weakly Luzin,  $T$  cannot be skinny, so  $\rho(T^*)$  is dense in  $S^{n-1}$ , so fix  $\xi \neq \eta$  such that  $\rho(x_\eta - x_\xi) \in W$ . There are then open  $U \ni x_\xi$  and  $V \ni x_\eta$  such that  $\rho(z - y) \in W$  for all  $y \in U$  and  $z \in V$ . Since  $|U \cap T| = |V \cap T| = \aleph_1$ , we may fix  $\alpha < \beta < \gamma$  with  $x_\alpha, x_\gamma \in U$  and  $x_\beta \in V$ ; then  $\rho(x_\beta - x_\alpha) \in W$  and  $\rho(x_\beta - x_\gamma) \in W$ , so  $p_\beta \in N_\alpha$  and  $p_\beta \in N_\gamma$ . ☕

Entangled subsets of  $\mathbb{R}$  were discussed by Avraham and Shelah [3] (see also [1]). The weakly Luzin sets and the entangled sets have a common generalization:

**Definition 2.6** For  $1 \leq n < \omega$  and  $1 \leq k < \omega$ :

1. If  $E \subseteq \mathbb{R}^n$ , then  $\tilde{E} \subseteq (\mathbb{R}^n)^k$  is derived from  $E$  iff  $\tilde{E} \subseteq E^k$  and whenever  $\vec{x} = \langle x_0, \dots, x_{k-1} \rangle \in \tilde{E}$  and  $\vec{y} = \langle y_0, \dots, y_{k-1} \rangle \in \tilde{E}$ :  $x_i \neq y_j$  unless  $i = j$  and  $\vec{x} = \vec{y}$ .
2.  $E$  is  $(n, k)$ -entangled iff  $E \subseteq \mathbb{R}^n$  is uncountable and whenever  $\tilde{E} \subseteq (\mathbb{R}^n)^k$  is uncountable and derived from  $E$ , and, for  $i < k$ ,  $W_i$  is open in  $S^{n-1}$  with  $W_i \neq \emptyset$ : there exist  $\vec{x}, \vec{y} \in \tilde{E}$  with  $\vec{x} \neq \vec{y}$  and  $\rho(x_i - y_i) \in W_i$  for all  $i$ .

Then “weakly Luzin” is equivalent to “ $(n, 1)$ -entangled”, and “ $k$ -entangled” is equivalent to “ $(1, k)$ -entangled”.  $E \subseteq \mathbb{R}$  is  $(1, 1)$ -entangled iff  $E$  is uncountable. If  $E$  is  $(n, k)$ -entangled and  $\tilde{E}$  and the  $W_i$  are as in (2), then there are actually uncountable disjoint  $X, Y \subseteq \tilde{E}$  such that  $\forall i \rho(x_i - y_i) \in W_i$  whenever  $\vec{x} \in X$  and  $\vec{y} \in Y$ . In (2), when  $k = 1$ , WLOG we may assume that  $W_0 = -W_0$ .

### 3 Preserving Failures of SOCA

The *Semi Open Coloring Axiom* (SOCA) is a well-known consequence of the PFA; see Abraham, Rubin, and Shelah [1]. We shall show that certain classes of failures of SOCA can be preserved in ccc extensions satisfying  $\text{MA} + 2^{\aleph_0} = \aleph_2$ . This is patterned after the proof (see [1, 3]) that an entangled set is consistent with  $\text{MA} + 2^{\aleph_0} = \aleph_2$ .

**Definition 3.1** For any set  $E$ : Let  $E^\dagger = (E \times E) \setminus \{(x, x) : x \in E\}$ . Fix  $W \subseteq E^\dagger$  with  $W = W^{-1}$ . Then  $T \subseteq E$  is  $W$ -free iff  $T^\dagger \cap W = \emptyset$  and  $T$  is  $W$ -connected iff  $T^\dagger \subseteq W$ .

**Definition 3.2**  $(E, W)$  is good iff  $E$  is an uncountable separable metric space,  $W = W^{-1}$  is an open subset of  $E^\dagger$ , and no uncountable subset of  $E$  is  $W$ -free.

Then, the SOCA is the assertion that whenever  $(E, W)$  is good, there is an uncountable  $W$ -connected set. An uncountable  $E \subseteq \mathbb{R}^n$  is weakly Luzin iff  $(E, W)$  is good for all  $W$  of the form  $\{(x, y) \in E^\dagger : \rho(x - y) \in A\}$ , where  $A \subseteq S^{n-1}$  is open and  $A = -A \neq \emptyset$ . We shall prove:

**Theorem 3.3** Assume that in the ground model  $\mathbf{V}$ ,  $\text{CH} + 2^{\aleph_1} = \aleph_2$  holds and  $E$  is a separable metric space. Then there is a ccc extension  $\mathbf{V}[G]$  satisfying  $\text{MA} + 2^{\aleph_0} = \aleph_2$  such that for all  $W \in \mathbf{V}$ , if  $(E, W)$  is good in  $\mathbf{V}$  then  $(E, W)$  is good in  $\mathbf{V}[G]$ .

A good  $(E, W)$  does not by itself contradict SOCA, since there may be an uncountable subset of  $E$  which is  $W$ -connected. But, if  $(E, U)$  and  $(E, W)$  are both good and  $U \cap W = \emptyset$ , then SOCA is contradicted, since any  $W$ -connected set is  $U$ -free. Such  $E, U, W$  are provided by a weakly Luzin  $E \subseteq \mathbb{R}^n$  (for  $n \geq 2$ ). The following combinatorial lemma will be used in the proof of Theorem 3.3.

**Lemma 3.4** *Assume the following:*

1. *CH holds.*
2.  *$m \in \omega$ ; and  $(E, W_i)$  is good for each  $i \leq m$ .*
3.  *$\theta$  is a suitably large regular cardinal and  $\langle M_\xi : \xi < \omega_1 \rangle$  is a continuous chain of countable elementary submodels of  $H(\theta)$ , with  $E \in M_0$  and each  $M_\xi \in M_{\xi+1}$ .*
4. *For  $x \in \bigcup_\xi M_\xi \setminus M_0$ :  $\text{ht}(x)$  is the  $\xi$  such that  $x \in M_{\xi+1} \setminus M_\xi$ .*
5.  *$x_\alpha^i \in E \setminus M_0$  for  $\alpha < \omega_1$  and  $i \leq m$ .*
6.  *$\text{ht}(x_\alpha^i) \neq \text{ht}(x_\beta^j)$  unless  $\alpha = \beta$  and  $i = j$ .*

*Then there are  $\alpha \neq \beta$  such that  $(x_\alpha^i, x_\beta^i) \in W_i$  for all  $i$ .*

We remark that (6) expresses the standard trick of using a set of points spaced by a chain of elementary submodels. In (5), we say  $x_\alpha^i \in E \setminus M_0$  so that  $\text{ht}(x_\alpha^i)$  is defined; note that by CH,  $E \subset \bigcup_\xi M_\xi$ .

**Proof.** Induct on  $m$ . When  $m = 0$ , this is immediate from the fact that  $(E, W_0)$  is good. Now, assume the lemma for  $m-1$ , and we prove it for  $m$ . Let  $\vec{x}_\alpha = \langle x_\alpha^0, \dots, x_\alpha^m \rangle \in E^{m+1}$ . Let  $\xi(\alpha, i) = \text{ht}(x_\alpha^i)$ . Thinning the  $\omega_1$ -sequence and rearranging each  $\vec{x}_\alpha$  if necessary, we may assume that  $\xi(\alpha, 0) < \xi(\alpha, 1) < \dots < \xi(\alpha, m)$  and that  $\alpha < \beta \rightarrow \xi(\alpha, m) < \xi(\beta, 0)$ . Let  $F = \text{cl}\{\vec{x}_\alpha : \alpha < \omega_1\} \subseteq E^{m+1}$ , and fix  $\mu < \omega_1$  such that  $F \in M_\mu$ ; there is such a  $\mu$  by CH.

For  $\alpha \geq \mu$ : Let  $K_\alpha = \{z \in E : \langle x_\alpha^0, \dots, x_\alpha^{m-1}, z \rangle \in F\}$ .  $K_\alpha$  is uncountable because  $K_\alpha \in M_{\xi(\alpha, m)}$  but  $K_\alpha$  contains the element  $x_\alpha^m \notin M_{\xi(\alpha, m)}$ . Since  $(E, W_m)$  is good, choose  $u_\alpha, v_\alpha \in K_\alpha$  with  $(u_\alpha, v_\alpha) \in W_m$ , and then choose disjoint basic open sets  $U_m, V_m \subseteq E$  with  $u_\alpha \in U_m, v_\alpha \in V_m$ , and  $(x, y) \in W_m$  for all  $x \in U_m$  and  $y \in V_m$ .

Of course,  $U_m, V_m$  depend on  $\alpha$ , but there are only  $\aleph_0$  possible choices, so fix an uncountable set  $I \subseteq \{\alpha : \mu \leq \alpha < \omega_1\}$  such that the  $U_m, V_m$  are the same for  $\alpha \in I$ . By the lemma for  $m-1$ , fix  $\gamma, \delta \in I$  such that  $\gamma \neq \delta$  and  $(x_\gamma^i, x_\delta^i) \in W_i$  for all  $i < m$ . Now choose disjoint open neighborhoods  $U_i$  of  $x_\gamma^i$  and  $V_i$  of  $x_\delta^i$  for  $i < m$  so that  $(x, y) \in W_i$  whenever  $x \in U_i$  and  $y \in V_i$ . Note that the two open sets  $\prod_{i \leq m} U_i$  and  $\prod_{i \leq m} V_i$  both meet  $F$ , since  $u_\gamma \in K_\gamma$  and  $v_\delta \in K_\delta$ , so  $\langle x_\gamma^0, \dots, x_\gamma^{m-1}, u_\gamma \rangle \in F \cap \prod_{i \leq m} U_i$  and  $\langle x_\delta^0, \dots, x_\delta^{m-1}, v_\delta \rangle \in F \cap \prod_{i \leq m} V_i$ . We may then choose  $\alpha, \beta$  such that  $\vec{x}_\alpha \in \prod_{i \leq m} U_i$  and  $\vec{x}_\beta \in \prod_{i \leq m} V_i$ . But then  $(x_\alpha^i, x_\beta^i) \in W_i$  for all  $i$ . ☕

**Lemma 3.5** *In the ground model  $\mathbf{V}$ : Assume CH, let  $(E, W)$  be good, and let  $\mathbb{Q}$  be any forcing poset such that  $q \Vdash_{\mathbb{Q}} "(E, W) \text{ is not good}"$  for some  $q \in \mathbb{Q}$ .*

*Then, in  $\mathbf{V}$ : there is a ccc poset  $\mathbb{P}$  of size  $\aleph_1$  such that  $\mathbb{Q} \times \mathbb{P}$  is not ccc and such that for all  $U \in \mathbf{V}$ : If  $(E, U)$  is good then  $\mathbb{1} \Vdash_{\mathbb{P}} "(E, U) \text{ is good}"$ .*

**Proof.** Extending  $q$ , we may assume that for some  $\mathbb{Q}$ -name  $\dot{Z}$ :  $q \Vdash "\dot{Z} \subseteq E$  is uncountable and  $W$ -free". Fix  $\theta$  and the  $M_\xi$  so that (3)(4) of Lemma 3.4 hold.

Now, inductively choose  $q_\alpha \leq q$  and  $x_\alpha^0, x_\alpha^1 \in E \setminus M_0$  for  $\alpha < \omega_1$  so that  $q_\alpha \Vdash x_\alpha^0, x_\alpha^1 \in \dot{Z}$  and such that  $\text{ht}(x_\alpha^0) < \text{ht}(x_\alpha^1) < \text{ht}(x_\beta^0)$  whenever  $\alpha < \beta < \omega_1$ . Let

$$\mathbb{P} = \{p \in [\omega_1]^{<\omega} : \forall \{\alpha, \beta\} \in [p]^2 [(x_\alpha^0, x_\beta^0) \in W \text{ or } (x_\alpha^1, x_\beta^1) \in W]\} \quad .$$

$\mathbb{P}$  is ordered by reverse inclusion, with  $1 = \emptyset$ . Each  $\{\alpha\} \in \mathbb{P}$ , and the pairs  $(q_\alpha, \{\alpha\}) \in \mathbb{Q} \times \mathbb{P}$  are incompatible, so  $\mathbb{Q} \times \mathbb{P}$  is not ccc.

Now, suppose that we have some good  $(E, U)$  and  $p \Vdash_{\mathbb{P}} "(E, U) \text{ is not good}"; we shall derive a contradiction. Extending  $p$ , we may assume that for some  $\mathbb{P}$ -name  $\dot{T}$ :  $p \Vdash "\dot{T} \subseteq E$  is uncountable and  $U$ -free". Then, inductively choose  $p_\mu \leq p$  and  $t_\mu \in E \setminus M_0$  for  $\mu < \omega_1$  so that  $p_\mu \Vdash t_\mu \in \dot{T}$  and such that  $\text{ht}(t_\mu) < \text{ht}(t_\nu)$  whenever  $\mu < \nu < \omega_1$ . Our contradiction will use the observation:$

$$\mu \neq \nu \rightarrow (t_\mu, t_\nu) \notin U \text{ or } p_\mu \perp p_\nu \quad . \quad (*)$$

Thinning the sequence and extending  $p$  if necessary, we may assume that the  $p_\mu$  form a  $\Delta$  system with root  $p$ ; so  $p_\mu = p \cup \{\alpha(0, \mu), \dots, \alpha(c, \mu)\}$ , with  $\alpha(0, \mu) < \dots < \alpha(c, \mu)$ . We also assume that  $\max(p) < \alpha(0, 0)$  and  $\mu < \nu \rightarrow \alpha(c, \mu) < \alpha(0, \nu)$ . Since  $p_\mu \in \mathbb{P}$ ,

$$i \neq j \rightarrow (x_{\alpha(i, \mu)}^0, x_{\alpha(j, \mu)}^0) \in W \text{ or } (x_{\alpha(i, \mu)}^1, x_{\alpha(j, \mu)}^1) \in W$$

for each  $\mu$ . Let  $\vec{x}_\mu = (x_{\alpha(0, \mu)}^0, x_{\alpha(0, \mu)}^1, \dots, x_{\alpha(c, \mu)}^0, x_{\alpha(c, \mu)}^1) \in E^{2(c+1)}$ . Since  $W$  is open, we may thin again and assume that all  $\vec{x}_\mu$  are sufficiently close to some condensation point of  $\{\vec{x}_\mu : \mu < \omega_1\}$  so that for all  $\mu, \nu$ :

$$i \neq j \rightarrow (x_{\alpha(i, \mu)}^0, x_{\alpha(j, \nu)}^0) \in W \text{ or } (x_{\alpha(i, \mu)}^1, x_{\alpha(j, \nu)}^1) \in W \quad .$$

Thus, if  $p_\mu \perp p_\nu$  then the incompatibility must come from the same index  $i$ , so that  $(*)$  becomes

$$\mu \neq \nu \rightarrow (t_\mu, t_\nu) \notin U \text{ or } \exists i \leq c [(x_{\alpha(i, \mu)}^0, x_{\alpha(i, \nu)}^0) \notin W \text{ and } (x_{\alpha(i, \mu)}^1, x_{\alpha(i, \nu)}^1) \notin W] \quad .$$

This comes close to contradicting Lemma 3.4. With an eye to satisfying hypothesis (6), we thin the sequence again and assume that  $\text{ht}(t_\mu) \neq \text{ht}(x_{\alpha(i, \nu)}^\ell)$  whenever  $\mu \neq \nu$ . It is still possible to have  $\text{ht}(t_\mu) = \text{ht}(x_{\alpha(i, \mu)}^\ell)$ , but for each  $\mu$ ,  $\text{ht}(t_\mu) = \text{ht}(x_{\alpha(i, \mu)}^\ell)$  can hold for at most one pair  $(\ell, i)$ . Thinning once more, we can assume WLOG that this  $\ell$  is always 1, so that  $\text{ht}(t_\mu) \neq \text{ht}(x_{\alpha(i, \nu)}^0)$  for all  $\mu < \omega_1$  and all  $i \leq c$ . But now the  $(c+2)$ -tuples  $(t_\mu, x_{\alpha(0, \mu)}^0, \dots, x_{\alpha(c, \mu)}^0)$  (for  $\mu < \omega_1$ ) contradict Lemma 3.4, where  $W_0 = U$  and the other  $W_i = W$ .

We also need to show that  $\mathbb{P}$  is ccc. If this fails, then choose the  $p_\mu$  to enumerate an antichain. Derive a contradiction as before, but replace  $(*)$  by the stronger fact  $\mu \neq \nu \rightarrow p_\mu \perp p_\nu$ , and delete all mention of  $\dot{T}$  and the  $t_\mu$ . ☕

We remark that a simplification of the above proof yields the standard proof that an instance of SOCA can be forced by a ccc poset. Forget about  $\mathbb{Q}$  and just assume

that  $(E, W)$  is good. Choose the  $x_\alpha \in E \setminus M_0$  for  $\alpha < \omega_1$  so that  $\text{ht}(x_\alpha) < \text{ht}(x_\beta)$  whenever  $\alpha < \beta < \omega_1$ .  $\mathbb{P}$  is now  $\{p \in [\omega_1]^{<\omega} : \forall \{\alpha, \beta\} \in [p]^2 [(x_\alpha, x_\beta) \in W]\}$ . Then some  $p \in \mathbb{P}$  forces an uncountable  $W$ -connected set.

**Proof of Theorem 3.3.** In the ground model  $\mathbf{V}$ , we build a normal chain of ccc posets,  $\langle \mathbb{F}_\alpha : \alpha \leq \omega_2 \rangle$ , where  $\alpha < \beta \rightarrow \mathbb{F}_\alpha \subseteq_c \mathbb{F}_\beta$  and we take unions at limits. So, our model will be the generic extension  $\mathbf{V}[G]$  given by  $\mathbb{F}_{\omega_2}$ .  $|\mathbb{F}_\alpha| \leq \aleph_1$  for all  $\alpha < \omega_2$ , while  $|\mathbb{F}_{\omega_2}| = \aleph_2$ . Given  $\mathbb{F}_\alpha$ , we choose  $\dot{\mathbb{P}}_\alpha$ , which is an  $\mathbb{F}_\alpha$ -name forced by  $\mathbb{1}$  to be a ccc poset of size  $\aleph_1$ ; then  $\mathbb{F}_{\alpha+1} = \mathbb{F}_\alpha * \dot{\mathbb{P}}_\alpha$ .

The standard bookkeeping which is used to guarantee that  $\mathbf{V}[G] \models \text{MA} + 2^{\aleph_0} = \aleph_2$  is modified slightly here, since we need to assume inductively that  $\mathbb{1} \Vdash_{\mathbb{F}_\alpha} "(E, W) \text{ is good}"$  for all  $W$  such that  $(E, W)$  is good in  $\mathbf{V}$ . This is easily seen (similarly to Theorem 49 of [6]) to be preserved at limit  $\alpha$ . For the successor stage, assume that we have  $\mathbb{F}_\alpha$  and the standard bookkeeping says that we should use  $\dot{\mathbb{Q}}_\alpha$ , which is an  $\mathbb{F}_\alpha$ -name which is forced by  $\mathbb{1}$  to be a ccc poset of size  $\aleph_1$ . Roughly, we ensure that either MA holds for  $\dot{\mathbb{Q}}_\alpha$  or  $\dot{\mathbb{Q}}_\alpha$  ceases to be ccc. More formally, choose  $\dot{\mathbb{P}}_\alpha$  as follows:

Consider this from the point of view of the  $\mathbb{F}_\alpha$ -extension  $\mathbf{V}[G \cap \mathbb{F}_\alpha]$ . In this model, CH holds, and we have a ccc poset  $\mathbb{Q}_\alpha$ , and we must define another ccc poset  $\mathbb{P}_\alpha$ . We know (using our inductive assumption) that for all  $W \in \mathbf{V}$ , if  $(E, W)$  good in  $\mathbf{V}$  then it is still good. If for all such  $W$ ,  $\mathbb{1} \Vdash_{\mathbb{Q}_\alpha} "(E, W) \text{ is good}"$ , then let  $\mathbb{P}_\alpha = \mathbb{Q}_\alpha$ . If not, then fix  $W \in \mathbf{V}$  with  $(E, W)$  good in  $\mathbf{V}$  such that  $q \Vdash_{\mathbb{Q}_\alpha} "(E, W) \text{ is not good}"$  for some  $q \in \mathbb{Q}_\alpha$ . Still working in  $\mathbf{V}[G \cap \mathbb{F}_\alpha]$ , we apply Lemma 3.5 and let  $\mathbb{P}$  be a ccc poset of size  $\aleph_1$  such that  $\mathbb{Q}_\alpha \times \mathbb{P}$  is not ccc and such that for all  $U \in \mathbf{V}[G \cap \mathbb{F}_\alpha]$  (and hence for all  $U \in \mathbf{V}$ ): If  $(E, U)$  is good then  $\mathbb{1} \Vdash_{\mathbb{P}} "(E, U) \text{ is good}"$ . Since  $\mathbb{Q}_\alpha \times \mathbb{P}$  is not ccc, we may fix  $p_0 \in \mathbb{P}$  such that  $p_0 \Vdash_{\mathbb{P}} "\mathbb{Q}_\alpha \text{ is not ccc}"$ . We cannot claim that  $\mathbb{1} \Vdash_{\mathbb{P}} "\mathbb{Q}_\alpha \text{ is not ccc}"$ , so let  $\mathbb{P}_\alpha = p \downarrow = \{p \in \mathbb{P} : p \leq p_0\}$ . Then  $\mathbb{1}_{\mathbb{P}_\alpha} = p \Vdash_{\mathbb{P}_\alpha} "\mathbb{Q}_\alpha \text{ is not ccc}"$ , and all good  $(E, U)$  from  $\mathbf{V}$  remain good in the  $\mathbb{P}_\alpha$  extension.

Now, in  $\mathbf{V}$ , let  $\dot{\mathbb{P}}_\alpha$  be the name for this  $\mathbb{P}_\alpha$  as chosen above. ☕

**Proof of Theorem 1.1.** In the ground model  $\mathbf{V}$ , assume that  $2^{\aleph_0} = \aleph_1$  and  $2^{\aleph_1} = \aleph_2$ . By CH, we may fix a (weakly) Luzin set  $E \subseteq \mathbb{R}^n$  (where  $n \geq 2$ ). Now, apply Theorem 3.3. ☕

## 4 Use of SOCA

It is easily seen directly that a weakly Luzin set contradicts SOCA, so that the Filippov space cannot exist under SOCA. Somewhat more generally,

**Theorem 4.1** *Assume SOCA. Let  $X$  be compact, with a continuous map  $\pi : X \rightarrow Y$ , where  $Y$  is compact metric. Assume further that there is an uncountable  $E \subseteq Y$  such that for  $y \in E$ , there are three points  $x_y^i \in \pi^{-1}\{y\}$  for  $i = 0, 1, 2$  and disjoint open neighborhoods  $U_y^i$  of  $x_y^i$  such that  $\pi(U_y^i) \cap \pi(U_y^j) = \{y\}$  whenever  $i \neq j$ .*

*Then  $X$  has an uncountable discrete subset.*

Note that the double arrow space satisfies these hypotheses with “three” weakened to “two”, while the Filippov space satisfies these hypotheses with “three” strengthened to “omega”.

**Proof.** Let  $F_y^i = \text{cl}(\pi(U_y^i))$ , which is a closed set in  $Y$  containing  $y$ . Shrinking the  $U_y^i$ , we may assume that the three sets  $F_y^i \setminus \{y\}$  are pairwise disjoint.

We use CSM, which is a consequence of SOCA; see [10]. Call  $T \subseteq E$  *i*-connected iff for all  $\{y, z\} \in [T]^2$ , either  $y \in F_z^i$  or  $z \in F_y^i$ . Call  $T$  *i*-free iff for all  $\{y, z\} \in [T]^2$ , both  $y \notin F_z^i$  and  $z \notin F_y^i$ . Applying CSM three times, we get an uncountable  $T \subseteq E$  such that for each  $i$ , either  $T$  is *i*-connected or  $T$  is *i* free. By the disjointness of the  $F_y^i \setminus \{y\}$ ,  $T$  can be *i*-connected for at most two values of  $i$ . Fixing  $i$  such that  $T$  is *i*-free, we see that  $\{x_y^i : y \in T\}$  is discrete. ☕

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